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Extension of two-dimensional sampling theory

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Abstract. This paper extends limiting results obtained for the sampling interval at zero and infinity when various sampling patterns other than the conventional rectangular grid are used.

1. Introduction

This paper is concerned with methods of representing real surfaces in digital form. The transition from analogue data obtained from surface measuring instruments to digital form is necessary to allow a wide degree of flexibility in estimating parameters. In this paper the surfaces of interest are those taken from machined parts. From such surfaces information can be obtained which is fundamental to the functional properties of components, especially those functions concerned with contact such as friction, wear and lubrication. In these contact situations the properties of the surface summits are critical. Typical properties of interest include the density of summits, their expected heights, curvatures and slopes.

Conventional digital representations of surfaces consist largely of individual one-dimensional (1D) profiles of the surface. In this case simple three-point analysis for peak definition is used; a peak exists if the two contiguous digital ordinates are lower than the central one. Further refinement can be achieved using more points but they are still in a straight line. A much more realistic representation of the surface is to use digital points on a two-dimensional (2D) plane. However, moving into the 2D plane brings with it a new degree of complexity; there are many different possibilities for defining a summit (2D peak). The cases for four and five points have already been studied by Whitehouse and Phillips (1985) but there are others such as the hexagonal case investigated here. Obviously more refinement of ordinary parameters such as curvature or slope can be catered for by using Lagrangian 2D differentiation (see Abramowitz and Stegun 1965) , but these only allow for rectangular grids and do not take into account many different definitions of summits.

Obviously there is a natural inclination to use more and more digital points in the modelling in order to get results which are closer to the theoretical continuous results. But a price has to be paid for these more comprehensive models. The penalty is usually in the form of data storage or processing time. It is the purpose of this paper to see if there are any advantages of using the hexagonal sampling scheme and the seven-point (k = 6) model for a summit as opposed to such other schemes as the digonal, the

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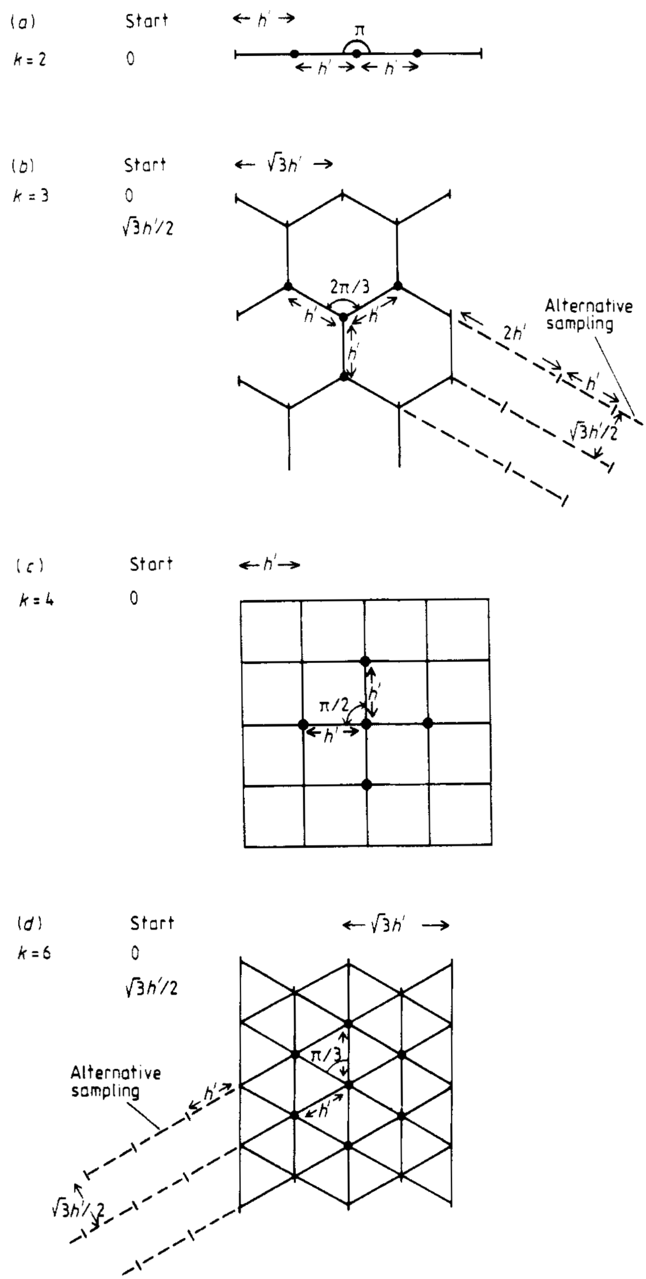

Figure 1. Different sampling schemes. (*a*) Three-point scheme (digonal) for $k = 2$. (*b*) Four-point scheme (trigonal) for $k = 3$ showing alternate start points and alternative sampling direction at 30°. (*c*) Five-point scheme (tetragonal) for $k = 4$. (*d*) Seven-point scheme (hexagonal) for $k = 5$ showing alternate start points and alternative sampling direction at 60°.

trigonal and in particular the rectangular (tetragonal) five-point ($k = 4$) model (see figure 1).

It has been pointed out by Preston (1979) that there is a reluctance in using anything other than a rectangular sampling scheme because it is perceptually more satisfactory for humans to observe straight vertical features on a square grid pattern. He also pointed out that robots would not have this same bias.

The obvious reasons, apart from those given above, for using the rectangular grid is that the transformation from 1D to 2D is more natural. A usual example pointed out is that of the fast Fourier transform (FFT). It is very simple to progress from a 1D FFT to a 2D FFT simply by taking all the rows in turn and then the columns. What is not so obvious, however, are the results obtained by Mersereau (1979), in robot vision applications, who showed that by using a hexagonal sampling scheme the 2D FFT required less computational time than when a rectangular grid was used, especially if the 2D image had circular symmetry. Similar advances occurred in filtering. Also it was found that 13.4% of data points were needed to cover an image yet preserving the same bandwidth. It is also true to say that hexagonal sampling is more symmetrical. Only three correlation coefficients are required between ordinates in the $k = 6$ case whereas in the rectangular case the number can be up to six different coefficients depending on the numerical model.

On the debit side it has to be admitted that hexagonal sampling schemes are more difficult to incorporate into TV scan systems because of the need to delay alternate scan lines by one half a sample interval and also that if a regular hexagonal geometry is required the sample interval has to be increased by a factor of $2/\sqrt{3}$. These combine to make programming and testing more difficult.

At present hexagonal sampling has been used principally in TV image processing particularly for robot vision. In this application, what are often being examined are edges and curves. There is considerable evidence (see Staunton 1989) that much improvement in processing speed can be obtained with little change to TV frame grab routines. Nearly 50% speed-up has already been achieved.

Unfortunately considerations applicable to flow and edge detection in vision systems are not necessarily useful in surface metrology although the technique has been used in detecting flows in sand moulds. In surface metrology for tribology it is the specific height (or intensity) information which is being sought, and its distribution. This is prone to much greater errors especially in the measurement of differentials to get curvature and slope as opposed to differentiating to enhance edges as in vision applications.

It is the purpose of this paper to bring together all the results suited for tribology obtained so far including all the existing knowledge as well as new information on the hexagonal sampling method. From this collation it is intended to make judgements on the complexity of model needed to achieve a result to within a known percentage of the theoretical continuous results.

2. Sampling schemes

A number of sampling schemes for sampling in a 2D plane were discussed by Whitehouse and Phillips (1985). These can best be visualised in figure 1 by means of a circle whose centre is an ordinate, and around the circumference of which are k evenly spaced ordinates, with angular spacing $2\pi/k$. Whitehouse and Phillips (1985) discussed the cases of $k = 2$ (for profile sampling) and $k = 3$ and 4 (for 2D plane sampling). The next step is to introduce the seven-point (hexagonal) case of $k = 6$, which has six evenly spaced ordinates around the circumference with angular spacing $\pi/3$. To produce an appropriate grid it would be necessary to sample along parallel lines in a manner similar to that which produced the grid for the four-point (trigonal) case of $k = 3$. The radius of the circle is assumed to be $h' = \Delta h$, i.e. this is the distance between the central ordinate and the surrounding ordinates. If the same unit of measurement were used for all schemes (values of k) then this would lead to different densities of ordinates (see table 1 of Whitehouse and Phillips 1985). However, to make comparisons between schemes more appropriate (and related to the information used) Δ is chosen for each scheme (value of k) to ensure that the density of ordinates is unity when h is unity. So $\Delta = 2/(27^{1/4})$ if $k = 3$, $\Delta = 1$ if $k = 4$ and $\Delta = 2^{1/2}/(3^{1/4})$ if $k = 6$. These schemes are used to measure peaks of the profile and summits of the surface. This is done by defining 'discrete' peaks and summits whenever a central ordinate Z_0 is higher (larger in magnitude) than the surrounding k ordinates.

3. Limiting results

Results for the distribution of the peak and summit height have been obtained by Rice (1945) and Nayak (1971) for the continuous Gaussian surface with zero mean and unit variance. Whitehouse and Phillips (1985) compared these results with those obtained with 'discrete' definitions using the discrete sampling schemes discussed in § 3, for $k = 2$, 3 and 4, as the sampling interval $h' = \Delta h$ converges to 0. These results will now be supplemented by the result for $k = 6$. To do this it is necessary to make assumptions about the behaviour near the origin of the autocorrelation function $\rho(t)$ for ordinates a distance t apart. It will be assumed that

$$\rho(t) = 1 + D_2\, t^2/2! + D_4\, t^4/4! + o(t^4) \tag{3.1}$$

where $D_2 < 0$, $D_4 > 0$ and

$$\eta = -D_2/\sqrt{D_4} < \sqrt{5/6}. \tag{3.2}$$

Whitehouse and Phillips (1985) considered the density of peaks and summits. The results for isotropic continuous Gaussian surfaces are known and were given for peaks as

$$D_{\text{peak}} = \sqrt{D_4/(-D_2)}/(2\pi) \tag{3.3}$$

by Rice (1945) and for summits as

$$D_{\text{sum}} = [D_4/(-D_2)]/(6\pi\sqrt{3}) \tag{3.4}$$

by Nayak (1971).

The density of peaks or summits is the number of peaks per unit length or summits per unit area, using the $(k+1)$-point definition of peak for $k = 2$ and of summit for $k = 3$, 4 and 6. The expected density of peaks or summits is the product of the $\text{pr}(T_{k+1})$ and the density of ordinates $1/h^2$, where T_{k+1} is the event $(S_1 > 0, S_2 > 0, \ldots, S_k > 0)$ and S_1 to S_k are the differences between the central ordinate and the k adjacent ordinates at a distance h'. This probability $\text{pr}(T_{k+1})$ is known as an orthant probability.

The limiting behaviours of the orthant probabilities $\text{pr}(T_{k+1})$ as h tends to 0 are given in table 1, for $k = 2$, 3, 4 and 6. The results for $k = 2$, 3 and 4 are taken from Whitehouse and Phillips (1985) though they have been adjusted for $k = 3$ to make the

Table 1. Expected summit (peak) density.

Model	k	$\mathrm{pr}(T_{k+1})$ limit as $h \to 0$	Expected density limit as $h \to 0$	Expected density limit as $h \to \infty$
Three points	2	$\frac{1}{2\pi}\left(\frac{D_4}{-D_2}\right)^{1/2} h$	$\frac{1}{2\pi}\left(\frac{D_4}{-D_2}\right)^{1/2} = D_{\mathrm{peak}}$	$\frac{1}{3h} = \frac{0.333}{h}$
Four points	3	$\frac{1}{6\pi}\left(\frac{D_4}{-D_2}\right) h^2 = 0.0531\left(\frac{D_4}{-D_2}\right) h^2$	$\sqrt{3}\, D_{\mathrm{sum}} = 1.732 D_{\mathrm{sum}}$	$\frac{1}{4h^2} = \frac{0.125}{h^2}$
Five points	4	$\frac{[\pi + 2\sin^{-1}(\frac{1}{3}) + 4\sqrt{2}]}{24\pi^2}\left(\frac{D_4}{-D_2}\right) h^2$ $= 0.400\left(\frac{D_4}{-D_2}\right) h^2$	$\frac{\sqrt{3}[\pi + 2\sin^{-1}(\frac{1}{3}) + 4\sqrt{2}]}{4\pi} D_{\mathrm{sum}}$ $= 1.306 D_{\mathrm{sum}}$	$\frac{1}{5h^2} = \frac{0.2}{h^2}$
Seven points	6	$\frac{[\pi + 6(\sqrt{3} - 1)]}{24\pi^2}\left(\frac{D_4}{-D_2}\right) h^2$ $= 0.0318\left(\frac{D_4}{-D_2}\right) h^2$	$\frac{\sqrt{3}[\pi + 6(\sqrt{3} - 1)]}{4\pi} D_{\mathrm{sum}}$ $= 1.038 D_{\mathrm{sum}}$	$\frac{1}{7h^2} = \frac{0.143}{h^2}$

density of ordinates $1/h^2$. In all cases the limit as h tends to zero is zero. However, the rate of convergence to zero is the important factor. It is seen that for the 1D profile ($k = 2$) the limit behaves as

$$C_2\sqrt{D_4/(-D_2)}\, h \tag{3.5}$$

and for the 2D plane ($k = 3$, 4 and 6) the limit behaves as

$$C_k[D_4/(-D_2)]\, h^2 \tag{3.6}$$

for suitable constants C_k. In this paper the new result for the case when $k = 6$ is given. The problem in obtaining this is that there are no general formulae for orthant probabilities for $k > 3$, except in special cases. To obtain C_6 the second derivative of the orthant probability is needed. This can be obtained by using the differential reduction formula of Plackett (1954) as the differential reduces the dimension of the orthant probability by two. When this method was applied two orthant probabilities of dimension two and four were obtained which could be evaluated. The mathematical details are given in the appendix.

The expected density of 'discrete' summits is the product of the orthant probability and $1/h^2$. Hence in the limits as h tends to 0 this expectation is half the second derivative of the orthant probability. These are also given in table 1. For profiles ($k = 2$) the expected density of peaks converges to the continuous result, D_{peak}, given by Rice (1945). For the 2D plane ($k = 3$, 4 and 6) the expected density of 'discrete' summits does not converge to the continuous results, D_{sum}, given by Nayak (1971). In all cases the limit is greater than D_{sum}, by 73% for $k = 3$, by 31% for $k = 4$ and by 4% for $k = 6$.

The limits as h tends to infinity are also given in table 1. In this case the autocorrelation function becomes zero and as the joint distribution of the ordinates is Gaussian the ordinates are independent. So the orthant probability is the probability that the central ordinate is greater than the k surrounding ordinates. It is well known that this probability is $1/(k+1)$, a result given by David and Six (1971). Hence the expected density of a peak behaves as $1/3h$ for $k = 2$ and the expected density of a summit behaves as $1/[(k+1)h^2]$ for $k = 3$, 4 and 6.

The expected peak or summit height of a surface with ordinates Z was also considered by Whitehouse and Phillips (1985). The results are known for isotropic continuous Gaussian surfaces and were given for peaks as

$$E(Z \mid \text{continuous peak}) = \sqrt{\pi/2}\,\eta \tag{3.7}$$

by Rice (1945) and for summits as

$$E(Z \mid \text{continuous summit}) = (4/\sqrt{\pi})\,\eta \tag{3.8}$$

by Nayak (1971). So the expected summit height for a 2D plane is 80% higher than the expected peak height on a profile.

Table 2. Expected summit (peak) height $E(Z_0 \mid T_{k+1})$.

Model	k	Expected summit (peak) height: limit as $h \to 0$	Expected summit (peak) height: limit as $h \to \infty$
Three points	2	$\left(\frac{\pi}{2}\right)^{1/2}\eta = 0.555\frac{4}{\sqrt{\pi}}\eta$	0.846
Four points	3	$2\left(\frac{3}{\pi}\right)^{1/2}\eta = 1.559\left(\frac{\pi}{2}\right)^{1/2}\eta = 0.866\frac{4}{\sqrt{\pi}}\eta$	1.029
Five points	4	$\frac{8\sqrt{2\pi}}{\pi + 2\sin^{-1}(\frac{1}{3}) + 4\sqrt{2}}\eta = 1.688\left(\frac{\pi}{2}\right)^{1/2}\eta = 0.938\frac{4}{\sqrt{\pi}}\eta$	1.163
Seven points	6	$\frac{3\sqrt{\pi}[6\pi - 12\tan^{-1}(1/\sqrt{2}) - \sqrt{2}\pi]}{\sqrt{\pi}[\pi + 6(\sqrt{3}-1)]}\eta = 1.779\left(\frac{\pi}{2}\right)^{1/2}\eta = 0.988\frac{4}{\sqrt{\pi}}\eta$	1.352

The limiting behaviours of the expected heights as h tends to 0 are given in table 2 for $k = 2$, 3, 4 and 6. The results for 2, 3 and 4 are taken from Whitehouse and Phillips (1985). In this paper the new result for $k = 6$ is given. The expected peak or summit height was given in the appendix of Whitehouse and Phillips (1982) for the general case of discrete measurements for the $(k+1)$-ordinate case with a radius h. For $k = 6$ this involves the ratio of two orthant probabilities of degree 5 in the the numerator and the orthant probability $\mathrm{pr}(T_{k+1})$ of degree 6 in the denominator. Now the behaviour of $\mathrm{pr}(T_{k+1})$ is known from table 1 so it is only necessary to ascertain the behaviour of the orthant probability in the numerator. It can be shown that the numerator behaves in the limit as

$$C_6'\sqrt{D_4/(-D_2)}\ h \tag{3.9}$$

for a suitable constant C_6'. The value of C_6' has been evaluated and the details are given in the appendix.

Using this result it is seen in table 2 that the limiting value of the expected summit height as h tends to 0 for $k = 6$ is $0.988(4\eta/\sqrt{\pi})$, i.e. almost 99% of the value of the expected height of the continuous surface given by Nayak (1971) in (3.8). So for profiles the expected height converges to the continuous result, $E(Z \mid \text{continuous peak})$, given by Rice (1945). For the 2D plane ($k = 3$, 4 and 6) the expected summit height does not converge to the continuous result, $E(Z \mid \text{continuous summit})$, given by Nayak (1971). In all cases the limit is smaller being 87% for $k = 3$, 94% for $k = 4$ and nearly 99% for $k = 6$.

The limits as h tends to infinity are also given in table 2. From the remarks above for these limits in table 1 it is seen that the expectations are the expected value of the largest ordinate in an independent sample of $(k + 1)$ Gaussian ordinates. These have been widely tabulated and the values that are given in table 2 can be found in David (1981, p 61).

4. Discussion

Results have been obtained for summit properties using different sampling schemes and their associated numerical models. It has been possible to extend the limiting results obtained by Whitehouse and Phillips (1985) to the seven-point (hexagonal) sampling case, for $k = 6$. This has been done by using the general methods given by Whitehouse and Phillips (1982) and Phillips (1984). The limiting results are obtained by using the reduction formula of Plackett (1954) which reduces the dimension of the orthant probabilities by 2. This reduction from six dimensions, for which there are no general formulae, to four dimensions enables the limiting behaviour of the orthant probabilities to be evaluated. The tedious details were checked with the computer algebra package REDUCE.

The object of the exercise was to determine the degree of complexity needed to approach the theoretical continuous results to within a satisfactory limit. This limit has been chosen to be ±5%. Figure 2 shows how the expected summit density and summit height compare with change in the numerical model. It is straightforward to obtain the expected summit curvatures from the formulae for expected summit heights given in Whitehouse and Phillips (1982, 1985). These three parameters are the most important ones in influencing surface properties.

As can be seen from figure 2 it is only when $k = 6$ (the hexagonal case) that the digital result for the expected summit density is within 5% of the continuous limit. So the hexagonal pattern is clearly needed in this case. However the situation as h tends to 0 for different models shows that the situation is far from clear for the expected summit height because even for $k = 4$ the result is very close to the 5% acceptable limit.

Another point to notice is the way in which the limiting results converge. When $k = 8$ the limiting results should equal the theoretical continuous results as h tends

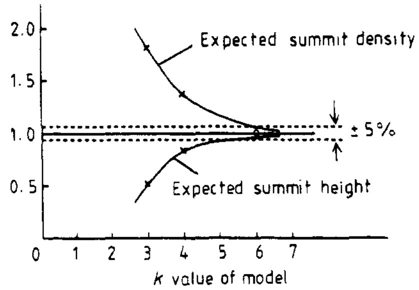

Figure 2. The diagram of expected summit density and expected summit height for different models (indicated by their k values) as $h \to 0$.

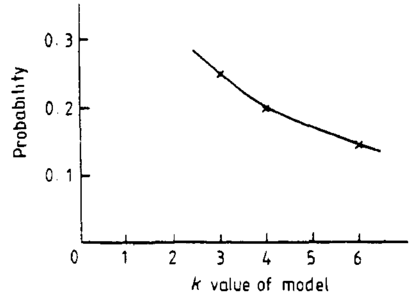

Figure 3. The probability of an ordinate being a summit as a function of k.

to 0 though this has still to be verified. It could be argued that the results should be extended to this case but the indications are that this is not practically necessary. In both cases the rate of change is much less dramatic as k gets large so that a law of diminishing returns is possible.

For the limiting case as h tends to infinity the probability that an ordinate is a summit is $1/(k+1)$ for model k and is shown in figure 3 as a function of k. As k tends to infinity this probability approaches zero.

The model results described in this paper for the new value of $k = 6$, the hexagonal model, represents a step forward in the digital representation of surfaces. Having this extra model has enabled the importance of digital models to be more fully appreciated apart from providing a more comprehensive coverage of the surface than was previously possible with just $k = 2$, 3 and 4. It is therefore a more robust digital procedure.

However, there are issues still to be resolved in the hexagonal case. It has been shown by Sharp (1961) that one advantage is that the data storage is 13% less than for the conventional rectangular case and that in many instances the processing is quicker. But these advantages are often apparent when the bandwidth of the signal has circular symmetry. Also because of this the processing speed-up favours edge enhancement and boundary tracking. Whether this benefit is maintained in the case of identifying summit properties remains still to be seen. Furthermore, surfaces which have centro-symmetrical bandwidth limitation are restricted to those which are isotropic. Such surfaces may be produced by shot blasting, electro discharge machining and some of the more modern processes such as ion-beam milling. Other processes like grinding and polishing are much more likely to have box-type bandwidth limitations which are more amenable to rectangular scanning schemes described by the model with $k = 4$.

There has been no comparison of the various sampling methods to a known surface. Prospective users of these methods need a measure of the advantages to be gained in accuracy of prediction of surface quality versus processing time for the different sampling schemes. Future work in this area, therefore, will be to perhaps examine other summit parameters and to examine the storage and processing times for surfaces having isotropic bandwidth limitations and which are anisotropic.

Appendix

Results for the probability density function and expectation of a 'discrete' peak (or summit) height can be obtained from the general results of truncated random variables given by Phillips (1984) and in the appendix of Whitehouse and Phillips (1982), and in particular for the peak height case by Whitehouse and Phillips (1978).

Define the m-dimensional orthant probability

$$\Phi^{(m)}(0;\mathbf{V}) = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \phi^{(m)}(\boldsymbol{x}';\mathbf{V})\,\mathrm{d}x_1 \ldots \mathrm{d}x_m \tag{A1}$$

where $\phi^{(m)}(\boldsymbol{x}';\mathbf{V})$ is the probability density function of a multivariate normal distribution for a random variable vector $\boldsymbol{X}$ of length m with zero means and variance–covariance matrix $\mathbf{V}$.

When $k = 6$ the probability that a central ordinate is a summit is given by the orthant probability

$$\mathrm{pr}(T_7) = \Phi^{(6)}(0;\mathbf{V}_6) \tag{A2}$$

with

$$\mathbf{V}_6 = \begin{pmatrix} 1 & \frac{1}{2} & b & c & b & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & b & c & b \\ b & \frac{1}{2} & 1 & \frac{1}{2} & b & c \\ c & b & \frac{1}{2} & 1 & \frac{1}{2} & b \\ b & c & b & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & b & c & b & \frac{1}{2} & 1 \end{pmatrix} \tag{A3}$$

where

$$b = (1 - 2\rho_1 + \rho_{\sqrt{3}})/(2 - 2\rho_1) \tag{A4}$$

$$c = (1 - 2\rho_1 + \rho_2)/(2 - 2\rho_1) \tag{A5}$$

$$\rho_t = \rho(t\Delta h) \tag{A6}$$

and

$$\Delta = 2^{1/2}/3^{1/4}. \tag{A7}$$

Using the reduction formula of Plackett (1954) the first derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}h}\Phi^{(6)}(0;\mathbf{V}_6) = B(h)\,\Phi^{(4)}(0;\mathbf{V}_4^{(b)}) + C(h)\,\Phi^{(4)}(0;\mathbf{V}_4^{(c)}). \tag{A8}$$

It is possible to show that

$$\lim \Phi^{(4)}(0;\mathbf{V}_4^{(b)}) = \tfrac{1}{24} \tag{A9}$$

$$B(h) \sim \frac{2}{\pi}\left(\frac{D_4}{-D_2}\right) h \tag{A10}$$

and

$$\lim C(h) = \frac{3}{2\pi}\left(\frac{D_4}{-D_2}\right)^{1/2} \tag{A11}$$

as h tends to 0 . However,

$$\lim \Phi^{(4)}(0; \mathbf{V}_4^{(c)}) = 0. \tag{A12}$$

To find out how this last orthant probability behaves as h tends to 0 we differentiate again. This leads to three orthant probabilities of dimension two, whose limits as h tends to 0 are 0.5, 0 and 0.5. Hence

$$\Phi^{(4)}(0; \mathbf{V}_4^{(c)}) \sim \frac{(\sqrt{3}-1)}{3\pi} \left(\frac{D_4}{-D_2}\right) h \tag{A13}$$

as h tends to 0. Putting all these results together gives

$$\Phi^{(6)}(0; \mathbf{V}_6) \sim \frac{\pi + 6(\sqrt{3}-1)}{24\pi^2} \left(\frac{D_4}{-D_2}\right) h^2 \tag{A14}$$

as h tends to 0.

The expected summit height (for $k = 6$) is given by

$$E(Z \mid T_7) = \frac{3\sqrt{(1-\rho_1)/\pi}\, \Phi^{(5)}(0; \mathbf{B}_6)}{\mathrm{pr}(T_7)} \tag{A15}$$

where $\mathbf{B}_6$ is the variance–covariance matrix of the conditional distribution of the differences $S_1, \ldots, S_5$ given S_6.

Now we have shown that $\mathrm{pr}(T_7) \sim \mathrm{O}(h^2)$ as h tends to 0 and from (3.1) that $\sqrt{1-\rho_1} \sim \mathrm{O}(h)$ as h tends to 0 so it is necessary to show that $\Phi^{(5)}(0; \mathbf{B}_6) \sim \mathrm{O}(h)$ as h tends to 0. This can be done by finding the derivative.

The matrix $\mathbf{B}_6$ has the form

$$\mathbf{B}_6 = \begin{pmatrix} 1 & B_1 & B_2 & B_3 & B_4 \\ B_1 & 1 & B_5 & B_6 & B_3 \\ B_2 & B_5 & 1 & B_5 & B_2 \\ B_3 & B_6 & B_5 & 1 & B_1 \\ B_4 & B_3 & B_2 & B_1 & 1 \end{pmatrix} \tag{A16}$$

where $B_1, \ldots, B_6$ are functions of b and c, and hence of h. It is therefore necessary to use the reduction formula of Plackett (1954) six times with $B_1, \ldots, B_6$. The reduction formula produces orthant probabilities of dimension three which can be evaluated by the well known formula given by David (1953). When these are combined they give

$$\Phi^{(5)}(0; \mathbf{B}_6) \sim \frac{(6\pi - 12\tan^{-1}(1/\sqrt{2}) - \sqrt{2}\pi)}{8\sqrt{3}\pi^2} \Delta \left(\frac{D_4}{-D_2}\right)^{1/2} h. \tag{A17}$$

Hence the result in table 2 for the limiting case as h tends to 0.

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